

PHYS402, Fall 2022

Lecture 9 Highlights

The Zeeman Effect

When a hydrogen atom is subjected to a strong external magnetic field $\vec{B}_{ext}$ the spectral emission lines are often observed to split. The splitting is roughly proportional to the strength of the magnetic field. Hale used this phenomenon to discover that sunspots are regions of strong magnetic field on the sun. Today magnetograms of the sun are taken frequently to document and quantify solar activity.

We can understand this Zeeman effect using perturbation theory applied to the Hydrogen atom.

The Hydrogen atom has a total magnetic moment made up of three contributions. The first is the orbital magnetic moment given by $\vec{\mu}_\ell = -\frac{e}{2m_e}\vec{L}$, where e is the electric charge and m_e is the mass of the electron. The second is the intrinsic magnetic moment of the electron, $\vec{\mu}_S = -\frac{e}{m_e}\vec{S}$, and the third is due to the magnetic moment of the nucleus $\vec{\mu}_N = \gamma_N\vec{S}_N$, where $\vec{S}_N$ is the nuclear spin and γ_N is the nuclear gyromagnetic ratio. Due to the large mass of the nucleus compared to the electron it is the case that $\gamma_N \ll \frac{e}{m_e}$ so we will ignore the nuclear magnetic moment in this calculation. In this approximation we now have $\vec{\mu}_{Total} = \vec{\mu}_\ell + \vec{\mu}_S = -\frac{e}{2m_e}(\vec{L} + 2\vec{S})$.

The external magnetic field will exert a torque on the total magnetic moment of the hydrogen atom. The Zeeman perturbing Hamiltonian is therefore $H_Z^1 = -\vec{\mu}_{Total} \cdot \vec{B}_{ext} = \frac{e}{2m_e}(\vec{L} + 2\vec{S}) \cdot \vec{B}_{ext}$. Note that this involves a new vector operator $\vec{L} + 2\vec{S} = \vec{J} + \vec{S}$. Note that the total magnetic moment of the atom is NOT parallel to $\vec{J}$, and this is a consequence of the non-classical properties of electron spin.

Now we have a situation in which there are two perturbations on the Hydrogen atom due to magnetic fields, namely $H = H^0 + H_{SO}^1 + H_Z^1$. How do we proceed from here? It depends upon which of the two perturbations is stronger.

Weak vs. Strong Field Zeeman Effect

Recall that the internal field in the Hydrogen atom is on the scale of $B_{int} \sim 10$ Tesla (Homework 5, problem 2). Therefore we will distinguish two limiting cases. The first is the weak Zeeman effect in which $B_{ext} \ll B_{int}$. In this case we can treat the unperturbed case as the spin-orbit corrected Hydrogen atom Hamiltonian and treat H_Z^1 as the perturbation. In the second case we have $B_{ext} \gg B_{int}$ and this is the strong Zeeman effect. In that case we will use the unperturbed Hydrogen atom as the starting point, ignore spin-orbit coupling, and treat the Zeeman Hamiltonian as the only perturbation.

Weak Field Zeeman Effect

In this case the internal field dominates, meaning that we start with the Hamiltonian for the Hydrogen atom of $H^0 + H_{SO}^1$ and the eigenfunctions in the coupled representation $|nljm_j\rangle$. The Zeeman Hamiltonian is the perturbation, leading to a first-order correction to the energy of $E_Z^1 = \langle nljm_j | H_Z^1 | nljm_j \rangle$. The states that we are beginning with are degenerate (the degeneracy of the unperturbed Hydrogen atom is $p = n^2$). Hence we

should be using degenerate perturbation theory here. However, with the choice of basis in the coupled representation the W matrix is diagonal, so there is no need to use degenerate perturbation theory since we have already found 'good' basis states.

The first order change in energy can be written as,

$$E_Z^1 = \frac{e}{2m_e} \vec{B}_{ext} \bullet \langle \vec{J} + \vec{S} \rangle.$$

Given the basis states it should be easy to calculate $\langle \vec{J} \rangle$, but what is $\langle \vec{S} \rangle$ in this case? We use the fact that both $\vec{S}$ and $\vec{L}$ are precessing rapidly due to the internal magnetic field (and resulting torque) in the Hydrogen atom, but their vector sum $\vec{J}$ is fixed. This means that $\langle \vec{S} \rangle$ will have no net component perpendicular to $\vec{J}$, at best only a component parallel to $\vec{J}$. This motivates us to write:

$$\langle \vec{S} \rangle = \langle \frac{\vec{S} \bullet \vec{J}}{|\vec{J}|^2} \vec{J} \rangle. \text{ Writing } \vec{L} = \vec{J} - \vec{S}, \text{ calculating } \vec{L} \bullet \vec{L} \text{ and solving for } \vec{S} \bullet \vec{J} \text{ yields,}$$

$$\langle \vec{S} \rangle = \frac{j(j+1) - \ell(\ell+1) + s(s+1)}{2j(j+1)} \langle \vec{J} \rangle.$$

Take the external magnetic field to define the z-direction: $\vec{B}_{ext} = B_{ext} \hat{z}$. The first order correction to the energy can be written as,

$$E_Z^1 = \mu_B g_J B_{ext} m_j, \text{ where}$$

$$\mu_B = \frac{e\hbar}{2m_e} \text{ is called the Bohr magneton and has a value of } 5.788 \times 10^{-5} \text{ eV/T,}$$

and $g_J = 1 + \frac{j(j+1) - \ell(\ell+1) + s(s+1)}{2j(j+1)}$ is called the Landé g-factor ($s = 1/2$ for the electron in the Hydrogen atom). Note that m_j typically has a range of positive and negative values, giving rise to a splitting of states depending on the z-component of $\vec{J}$. Note that this calculation predicts a splitting of states that is proportional to the external magnetic field strength.

Examples of the weak-field Zeeman effect in Hydrogen

Consider the Hydrogen atom in the $n = 2, \ell = 0, j = 1/2$ state, labeled $2s_{1/2}$. In this case the Hydrogen atom can have $m_j = +1/2$ or $-1/2$. The Landé g-factor evaluates to $g_J = 2$, so the Zeeman perturbed energies are $E_{Z,2s_{1/2}}^1 = \mu_B 2 B_{ext} (\pm 1/2) = \pm \mu_B B_{ext}$, giving rise to one positive energy and one negative energy perturbation.

Consider a second Hydrogen atom state: $n = 2, \ell = 1, j = 3/2$ state, labeled $2p_{3/2}$. In this case one finds $E_{Z,2p_{3/2}}^1 = \frac{4}{3} \mu_B B_{ext} m_j$, where m_j could be $(-\frac{3}{2}, -\frac{1}{2}, +\frac{1}{2}, +\frac{3}{2})$. This gives rise to 4 distinct states in finite field.

If we now consider transitions between these states it turns out that there are a total of 3 distinct wavelengths of light. Their splitting in energy is proportional to external magnetic field.

Strong Field Zeeman Effect

In this case the external field is much stronger than the internal field in the Hydrogen atom. In this case there is a strong torque on $\vec{J}$ and $\vec{J} + \vec{S}$, causing both to precess quickly. The spin-orbit effect is now dominated by the external field, meaning that the original un-coupled representation is better suited as the unperturbed starting point. We go back to the original unperturbed eigenfunctions in the un-coupled representation with $\psi^0 \propto |\ell m_\ell\rangle |s m_s\rangle$. Once again, this choice of basis diagonalizes the W matrix and there is no need to carry out a degenerate perturbation theory calculation.

The first order change in energy is now $E_{Z,strong}^1 = \frac{e}{2m_e} \vec{B}_{ext} \bullet \langle \vec{L} + 2\vec{S} \rangle$. Again taking the external magnetic field to define the z-direction: $\vec{B}_{ext} = B_{ext} \hat{z}$ we find $E_{Z,strong}^1 = \frac{e}{2m_e} B_{ext} \langle L_z + 2S_z \rangle$. In this basis we simply get the z-component quantum numbers, $E_{Z,strong}^1 = \mu_B B_{ext} (m_\ell + 2m_s)$. The perturbed energies are now, $E_{n,\ell,m_\ell,s,m_s} = -13.6 \text{ eV} / n^2 + \mu_B B_{ext} (m_\ell + 2m_s)$, where the sum of $m_\ell + 2m_s$ could be positive or negative.

Intermediate Field Zeeman Effect

In this case the two effects are of comparable size, and the perturbing Hamiltonian is given by $H' = H_{SO}^1 + H_{Rel}^1 + H_Z^1$. Now one must use degenerate perturbation theory since there is no basis in which this perturbation diagonalizes the W matrix.

At this point one has to choose a specific set of degenerate states, construct the corresponding W matrix with this perturbing Hamiltonian, and then diagonalize it to find the 'good' basis states. This is done for the case of $n = 2$ in Hydrogen, giving rise to an 8×8 W matrix (4 (ℓ, m_ℓ) states $\times$ 2 for spin). This calculation is carried out on pages 309-310 of Griffiths.